Найдите с.з и с.ф

$$\begin{cases} \Delta u(r,\varphi,z) = -\lambda u(r,\varphi,z), & 0<r<r_0,\ 0<\varphi<2\pi,\ 0<z<l \\ u(r_0,\varphi,z)=0 \\ u(r,\varphi,l)=0 \\ u(r,\varphi,0)=0 \end{cases}$$

$$\Delta_3 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial z^2}$$

1) $u = W(r,\varphi)\cdot Z(z)$

$$\Delta_2 W\cdot Z + W\cdot Z'' = -\lambda W\cdot Z$$

$$\underbrace{\frac{\Delta_2 W}{W}}_{=-\alpha} + \frac{Z''}{Z} = -\lambda$$

$$\frac{\Delta_2 W}{W} = -\alpha$$

2) $W = R(r)\cdot\Phi(\varphi)$

$$\frac{1}{r}(rR''+R')\Phi + \frac{1}{r^2}\Phi''R = -R\Phi\alpha$$

$$\frac{r^2R''+R'r}{R} + r^2\alpha + \underbrace{\frac{\Phi''}{\Phi}}_{=-\nu} = 0$$

3) $\begin{cases} \Phi''+\nu\Phi=0 \\ \Phi(\varphi+2\pi)=\Phi(\varphi) \end{cases}$

$\Phi_n = A_n\sin n\varphi + B_n\cos n\varphi$

$\nu_n = n^2 \qquad n=1,2,\ldots$

$\Phi_0 = A_0$

$\nu_n = 0$

4) $\begin{cases} r^2R''+rR'+R(r^2\alpha-n^2)=0 \\ R(r_0)=0 \end{cases}$

$R_{nk} = J_n\left(\frac{\mu_n^k r}{r_0}\right)$

$\mu_n^k$ - корень ур-ия $J_n(x)=0$

$\alpha_{n,k} = \left(\frac{\mu_n^k}{r_0}\right)^2$

5) $\begin{cases} Z''=(\alpha-\lambda)Z \\ Z(0)=Z(l)=0 \end{cases}$

$Z_m = \sin\left(\frac{m\pi}{l}z\right)$

$(\alpha-\lambda) = \left(\frac{m\pi}{l}\right)^2$

$$\lambda_{nkm} = \alpha - \left(\frac{m\pi}{l}\right)^2 = \left(\frac{\mu_n^k}{r_0}\right)^2 - \left(\frac{m\pi}{l}\right)^2$$

$$u_{nkm} = (A_n\sin n\varphi + B_n\cos n\varphi)\cdot J_n\left(\frac{\mu_n^k r}{r_0}\right)\cdot\sin\left(\frac{m\pi}{l}z\right)$$

$$u_{0km} = J_0\left(\frac{\mu_0^k r}{r_0}\right)\cdot\sin\left(\frac{m\pi}{l}z\right)$$

$$\lambda_{0km} = \left(\frac{\mu_0^k}{r_0}\right)^2 + \left(\frac{m\pi}{l}\right)^2$$

ответ

$$\begin{cases} \Delta_3 u + cu = F_0 = \text{const} & 0 \le r < a \\ u\big|_{r=a} = 0 \end{cases}$$

сфер сим $\Rightarrow$ $\Delta_3 = \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)$

Сделаем замену $u = \frac{v}{r}$ $\qquad \Delta_3\left(\frac{v}{r}\right) = \frac{v''}{r}$

$$\Rightarrow \frac{v''}{r} + c\frac{v}{r} = F_0 \Rightarrow \begin{cases} v'' + cv = F_0 r \\ v(a) = 0 \\ v(0) = 0 \end{cases}$$

1) $c = 0$

$$v'' = F_0 r$$
$$v' = F_0\frac{r^2}{2} + A$$
$$v = F_0\frac{r^3}{6} + Ar + B$$

$$v(0) = B = 0$$
$$v(a) = F_0\frac{a^3}{6} + Aa = 0 \Rightarrow F_0a^2 + 6A = 0$$
$$A = -\frac{F_0a^2}{6}$$

$$\Rightarrow v = \frac{F_0r^3}{6} - \frac{F_0a^2}{6}r \Rightarrow \boxed{u = \frac{F_0r^2}{6} - \frac{F_0a^2}{6} \quad \text{при } c = 0}$$

2) $c < 0$ $\qquad c = -\alpha^2$

$$v'' - \alpha^2 v = 0 \Rightarrow v = Ae^{\alpha r} + Be^{-\alpha r}$$

$v'' - \alpha^2 v = F_0 r$, частное решение $v = \frac{F_0 r}{-\alpha^2}$

$$\Rightarrow v = Ae^{\alpha r} + Be^{-\alpha r} + \frac{F_0 r}{-\alpha^2}$$

$$v(0) = A + B = 0$$
$$v(a) = Ae^{\alpha a} - Ae^{-\alpha a} + \frac{F_0 a}{-\alpha^2} = A(e^{\alpha a} - e^{-\alpha a}) + \frac{F_0 a}{-\alpha^2}$$

$$\Rightarrow A = \frac{F_0 a}{\alpha^2(e^{\alpha a} - e^{-\alpha a})} = \frac{F_0 a}{\alpha^2 2\operatorname{sh}(\alpha a)}$$

$$\Rightarrow v = \frac{F_0 a}{\alpha^2}\cdot\frac{\operatorname{sh}(\alpha r)}{\operatorname{sh}(\alpha a)} - \frac{F_0 r}{\alpha^2} \Rightarrow \boxed{u = \frac{F_0}{\alpha^2}\left(\frac{a}{r}\frac{\operatorname{sh}(\alpha r)}{\operatorname{sh}(\alpha a)} - 1\right) \quad \text{при } c < 0,\ c = -\alpha^2}$$

3) $c > 0$ $\qquad c = \alpha^2$

$v = A\sin\alpha r + B\cos\alpha r$ – общее реш, $v = \frac{F_0 r}{\alpha^2}$ – частное решение

$$v = A\sin\alpha r + B\cos\alpha r + \frac{F_0 r}{\alpha^2}$$

$$v(0) = B = 0$$
$$v(a) = A\sin\alpha a + F_0 a/\alpha^2$$

$$\Rightarrow v = -\frac{F_0 a}{\alpha^2\sin\alpha a}\cdot\sin\alpha r + \frac{F_0 r}{\alpha^2} \Rightarrow \boxed{u = \frac{F_0}{\alpha^2}\left(-\frac{a}{r}\frac{\sin\alpha r}{\sin\alpha a} + 1\right) \quad \text{при } c > 0,\ c = \alpha^2}$$

а) $\begin{cases} \Delta u - k^2 u = 0, & 0 < r < R \\ u|_{r=R} = A \end{cases}$

$\exists\, u = u(r)$

Замена $u = \frac{v}{r}$ $\Rightarrow$ $\begin{cases} v'' - k^2 v = 0 \\ v(R) = AR \\ v(0) = 0 \end{cases}$

$v = Ce^{kr} + Be^{-kr}$

$v(0) = C + B = 0$

$v(R) = Ce^{kR} - Ce^{-kR} = AR$

$\Rightarrow C = \frac{AR}{2\,sh(kR)}$

$\Rightarrow v = AR\,\frac{sh(kr)}{sh\,kR}$ $\Rightarrow$ $\boxed{u = A\,\frac{R}{r}\,\frac{sh(kr)}{sh(kR)}}$

б) $\begin{cases} \Delta u - k^2 u = 0 & r > R \\ u|_{r=R} = A \\ u \underset{r\to\infty}{\rightrightarrows} 0 \end{cases}$

$\exists\, u = u(r)$

Замена $u = \frac{v}{r}$ $\Rightarrow$ $\begin{cases} v'' - k^2 v = 0 \\ v(R) = AR \end{cases}$

$v = Ce^{kr} + Be^{-kr}$

т.к. $u = \frac{v}{r} = \frac{C}{r}e^{kr} + \frac{B}{r}e^{-kr} \underset{r\to\infty}{\rightrightarrows} 0$, то $C = 0$

$Be^{-kR} = AR \Rightarrow B = ARe^{kR}$

$\Rightarrow \boxed{u = \frac{AR}{r}e^{kR-kr}}$